White noise approach to the low density limit of a quantum particle in a gas

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Abstract

The white noise approach to the investigation of the dynamics of a quantum particle interacting with a dilute and in general non-equilibrium gaseous environment in the low density limit is outlined. The low density limit is the kinetic Markovian regime when only pair collisions (i.e., collisions of the test particle with one particle of the gas at one time moment) contribute to the dynamics. In the white noise approach one first proves that the appropriate operators describing the gas converge in the sense of appropriate matrix elements to certain operators of quantum white noise. Then these white noise operators are used to derive quantum white noise and quantum stochastic equations describing the approximate dynamics of the total system consisting of the particle and the gas. The derivation is given *ab initio*, starting from the exact microscopic quantum dynamics. The limiting dynamics is described by a quantum stochastic equation driven by a quantum Poisson process. This equation then applied to the derivation of quantum Langevin equation and linear Boltzmann equation for the reduced density matrix of the test particle. The first part of the paper describes the approach which was developed by L. Accardi, I.V. Volovich and the author and uses the Fock-antiFock (or GNS) representation for the CCR algebra of the gas. The second part presents the approach to the derivation of the limiting equations directly in terms of the correlation functions, without use of the Fock-antiFock representation. This approach simplifies the derivation and allows to express the strength of the quantum number process directly in terms of the one-particle S-matrix.

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1 Introduction

The fundamental equations in quantum theory are the Heisenberg and Schrödinger equations. However, it is a very difficult problem to solve explicitly these equations for realistic physical models and one uses various approximations or limiting procedures such as weak coupling and low density limits. These scaling limits describe the long time behavior of physical systems in different physical regimes.

One of the methods to study the long time behavior in quantum theory is the stochastic limit method, which was developed by Accardi, Lu and Volovich in the book [1]. The book is devoted mainly to the weak coupling regime, where one considers the long time dynamics of a quantum open system weakly interacting with a reservoir. The dynamics of the total system in this limit is described by the solution of a quantum stochastic differential equation driven by a quantum Brownian motion; the reduced dynamics of the system is described by a quantum Markovian master equation. Also in the low density regime the dynamics is described by a quantum stochastic differential equation. In this regime one considers the long time quantum dynamics of a test particle interacting with a dilute gas (we consider Bose gas in this paper) in the case the interaction is not weak but the density n of particles of the gas is small. The *low density limit* is the limit as $n \to 0$, $t \to \infty$, $nt = Const$ (t denotes time) of certain quantities such as matrix elements of the evolution operator or the reduced density matrix. In this limit the reduced time evolution for the test particle will be Markovian, since the characteristic time t_S for appreciable action of the gas on the test particle (time between collisions) is much larger than the characteristic time t_R for relaxation of correlations in the gas.

Accardi and Lu [2, 3, 4] and later Rudnicki, Alicki, and Sadowski [5] proved that the matrix elements of the evolution operator in the "collective vectors" converge in the low density limit to matrix elements of the solution of a quantum stochastic differential equation driven by a quantum Poisson process. The quantum Poisson process, introduced by Hudson and Parthasarathy [6], should arise naturally in the low density limit, as conjectured by Frigerio and Maassen [7].

The stochastic golden rule for the low density limit, that is a set of simple rules for the derivation of the limiting quantum white noise and stochastic differential equations, was developed in Refs [8, 9, 10], where the case of a discrete spectrum of the free test particle's Hamiltonian was considered. This method is based on the white noise approach and uses the stochastic limit technique [1] (in Ref. [11] the method is generalized to the case of a continuous spectrum). As the main result, the normally ordered quantum white noise and stochastic differential equations were derived. Then these equations were applied to the derivation of the quantum Langevin and master equations describing the evolution of the test particle.

The idea of the white noise approach is based on the fact that the rescaled free evolution of appropriate operators of the gas converges, in the sense of convergence of correlation functions, to certain operators of quantum white noise. Then, using these limiting operators which are called master fields, one can derive the quantum white noise equation for the limiting dynamics and put this equation to the normally ordered form which is equivalent to a quantum stochastic differential equation.

The white noise approach to the derivation of the stochastic equations in Refs. [2, 3, 4, 8, 9] uses the Fock-antiFock representation for the canonical commutation relations (CCR) algebra of the Bose gas, which is unitary equivalent to Gel'fand-Naimark-Segal (GNS) representation. The white noise approach of Ref. [10] considerably simplifies the calculations by avoiding use of the Fock-antiFock representation. A useful tool is the energy representation introduced in Refs. [8, 9], where the case of orthogonal formfactors was considered. This consideration was extended in Ref. [10] to the case of arbitrary formfactors and arbitrary, not necessarily equilibrium, quasifree low density states of the gas.

In Ref. [10] to each initial low density state of the Bose gas in the low density limit one associates a special "state", which is called a causal state, on the limiting master field algebra. The time-ordered (or causal) correlators of the initial Bose field converge to the causal correlators of the master fields, which are number operators of quantum white noise. This approach allows to express the intensity of the quantum Poisson process directly in terms of the one-particle S-matrix. In this case the algebra of the master fields (22), the limiting equation (30), and the quantum Ito table (32) do not depend on the initial state of the Bose gas (see Sect. 4 for details). Instead, the information on the initial state of the gas is contained in the limiting state φ_L of the master field [defined by (25)–(28)], which is now not the vacuum. Here the operator L determines the initial density of particles of the gas; if the gas is at equilibrium with inverse temperature β, then $L = e^{-\beta H_1}$ (see the next section). To get the master equation one has to take the conditional expectation with respect to the state φ_L.

The dynamics in the low density limit is given by the solution of the quantum white noise equation (30) or of the equivalent quantum stochastic equation

$$\mathrm{d}U_t = \mathrm{d}N_t(S-1)U_t \tag{1}$$

where U_t is the evolution operator at time t, S the one-particle S matrix describing scattering of the test particle on one particle of the gas, and $N_t(S-1)$ the quantum number process with strength $S-1$. In order to describe these objects let us introduce two Hilbert spaces $\mathcal{H}_{\mathrm{S}}$ and $\mathcal{H}_1$, which are called in this context the system and one-particle reservoir Hilbert spaces, and the Fock space $\Gamma(L^2(\mathbb{R}_+;\mathcal{H}_1))$ over the Hilbert space of square-integrable measurable vector-valued functions from $\mathbb{R}_+ = [0,\infty)$ to $\mathcal{H}_1$. Then the solution of the equation is a family of operators $U_t; t \geq 0$ in $\mathcal{H}_{\mathrm{S}} \otimes \Gamma(L^2(\mathbb{R}_+;\mathcal{H}_1))$ (adapted process); S is a unitary operator in $\mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_1$.

Let us define the number process. Let X be a self-adjoint operator in a Hilbert space $\mathcal{K}$; for any $f \in \mathcal{K}$ let $\Psi(f)$ be the normalized coherent vector in the Fock space $\Gamma(\mathcal{K})$. The *number operator* $N(X)$ is the generator of the one-parameter unitary group $\Gamma(e^{itX})$ characterized by $\Gamma(e^{itX})\Psi(f) = \Psi(e^{itX}f)$; $t \in \mathbb{R}$. The number operator is characterized by the property $\langle\Psi(f), N(X)\Psi(g)\rangle = \langle f, Xg\rangle\langle\Psi(f),\Psi(g)\rangle$. The definition of $N(X)$ is extended by complex linearity to any bounded operator X on $\mathcal{K}$. Let us consider $\mathcal{K}$ of the form $L^2(\mathbb{R}_+;\mathcal{H}_1) \cong L^2(\mathbb{R}_+) \otimes \mathcal{H}_1$. For any bounded operators $X_0 \in B(\mathcal{H}_{\mathrm{S}})$, $X_1 \in B(\mathcal{H}_1)$ and for any $t \geq 0$ define $N_t(X_0 \otimes X_1) := X_0 \otimes N(\chi_{[0,t]} \otimes X_1)$ and extend this definition by linearity to any bounded operator K in $\mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_1$. The family $\{N_t(K)\}_{t\geq 0}$ of operators in $\mathcal{H}_{\mathrm{S}} \otimes \Gamma(L^2(\mathbb{R}_+;\mathcal{H}_1))$ is called *quantum number process with strength* K.

Equations (1) and (16) describe the dynamics of the total system and can be applied, in particular, to the derivation of the irreversible quantum linear Boltzmann equation for the reduced density matrix of the test particle. This equation can be directly obtained from the quantum Langevin equation [9].

The reduced dynamics of the test particle in the low density limit with methods, based on a quantum Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy, has been investigated by Dümcke [12], where it is proved that, under some conditions, the reduced dynamics is given by a quantum Markovian semigroup.

In the white noise approach the reduced dynamics can be directly derived from the solution of the limiting quantum stochastic differential equation. Namely, the limiting evolution operator U_t and the limiting state φ_L determine the reduced dynamics by

$$T_t(X) = \varphi_L(U_t^+(X \otimes 1)U_t), \tag{2}$$

where X is any observable of the test particle, $\varphi_L(\cdot)$ denotes the conditional expectation, and T_t is the limiting semigroup. This equality shows that U_t is a stochastic dilation of the limiting Markovian semigroup. Using the quantum Ito table for stochastic differential $\mathrm{d}N_t$ one can derive a quantum Langevin equation for the quantity $U_t^+(X \otimes 1)U_t$. Then, taking partial expectation, one gets an equation for $T_t(X)$ and obtains the generator of the semigroup (see the end of Sec. 4). This is a general rule of the white noise approach: one first obtains the Langevin equation and then gets the reduced dynamics of the test particle. Let us note that although the quantum stochastic equation (16), which was derived in Refs. [2, 9], is different from (1) it gives the same reduced dynamics.

The low density limit can be applied to the model of a test particle moving through an environment of randomly placed, infinitely heavy scatterers (Lorentz gas) (see the review of Spohn[13]). In the Boltzmann–Grad limit successive collisions become independent and the averaged over the positions of the scatterers the position and velocity distribution of the particle converges to the solution of the linear Boltzmann equation. An advantage of the stochastic limit method is that it allows us to derive equations not only for averaged over reservoir degrees of freedom dynamics of the test particle but for the total system+reservoir. The convergence results and derivation of the linear Boltzmann equation for a quantum Lorentz gas in the low density and weak coupling limits are presented in Refs. [14, 15]. The Coulomb gas at low density is considered in Ref. [16].

The main results presented in this paper are: the causally normally ordered quantum white noise equations (16), (29) and equivalent quantum stochastic equation (36) for the limiting evolution operator; the quantum Langevin equation (17) for the evolution of any test particle's observable; the linear Boltzmann equation for the reduced density matrix (Theorem 4).

The structure of the paper is the following. In Sec. 2 a test particle interacting with a Bose gas is considered. In Sec. 3 the white noise approach developed by L. Accardi, I. Volovich, and the author in Refs. [8, 9] is presented. Sec. 4 describes the white noise approach developed in Ref. [10].

2 Test Particle Interacting with a Dilute Bose Gas

Consider two non-relativistic particles, with masses M and m, which are called the test particle and a particle of the gas. Suppose the particles interact by a pair potential $U(R-r)$, where R and r denote positions of particles. Then the classical dynamics of the particles is determined by the Hamiltonian $H_{\rm cl} = P^2/2M + p^2/2m + U(R-r)$, where P and p are momentums of the particles.

The quantum Hamiltonian of such system is obtained by identification of P with the momentum operator $\hat{P}$, R with the position operator $\hat{Q}$ and, if instead of one particle of mass m there is gas of these particles, by second quantization of the particles of the gas. This Hamiltonian has the form $H = H_S + H_R + H_{\rm int}$, where $H_S + H_R =: H_0$ is the free Hamiltonian with $H_S = \hat{P}^2/2M$, $H_R = \int \omega(p)a^+(p)a(p)dp$, $\omega(p) = p^2/2m$; the interaction Hamiltonian is $H_{\rm int} = \int U(\hat{Q}-r)a^+(r)a(r)dr$. The free Hamiltonian of one particle of the gas H_1 is the multiplication operator by the function ω. Boson annihilation and creation operators $a(k), a^+(p)$ satisfy the canonical commutation relations

$$[a(k), a^+(p)] = \delta(k-p)$$

The coordinate representation for these operators is introduced as $a(r) = \int e^{ikr}a(k)dk$. The interaction Hamiltonian can be written using the Fourier transform of the interaction potential $\tilde{U}(p) := \int U(r)e^{ipr}dr$ as

$$H_{\rm int} = \int dkdp\tilde{U}(p)e^{ip\hat{Q}} \otimes a^+(k-p)a(k) \tag{3}$$

This Hamiltonian acts in the Hilbert space $\mathcal{H}_{\rm S} \otimes \Gamma(\mathcal{H}_1)$, where $\mathcal{H}_{\rm S} = \mathcal{H}_1 = L^2(\mathbb{R}^3)$.

Important features of this system are that the interaction Hamiltonian $H_{\rm int}$ quadratic in creation and annihilation operators and commutes with the number operator, i.e., it preserves the number of particles of the gas.

We will consider, instead of (3), a different interaction Hamiltonian, which nevertheless keeps its basic properties. More precisely, we consider Hamiltonians of the form

$$H_{\rm int} := D \otimes A^+(g_0)A(g_1) + D^+ \otimes A^+(g_1)A(g_0)$$

where D is a bounded operator in $\mathcal{H}_{\rm S}$; $g_0, g_1 \in \mathcal{H}_1$ are two form-factors, and $A(g_0) = \int dkg_0^*(k)a(k)$ is the smeared annihilation operator. This Hamiltonian is also quadratic in creation and annihilation operators and preserves the number of particles of the gas. In the present paper we consider the case of a discrete spectrum for the free test particle's Hamiltonian, so that

$$H_{\rm S} = \sum_n \varepsilon_n P_n \tag{4}$$

where ε_n is an eigenvalue and P_n is the corresponding projector. This corresponds to the situation when the test particle is confined in some spatial region.

Density of particles of the gas is encoded in the state of the gas, which is chosen to be either the Gibbs state at inverse temperature β, chemical potential μ, and fugacity

$\xi = e^{\beta\mu}$, or a more general non-equilibrium Gaussian state, i.e., a gauge invariant mean zero Gaussian state with the two point correlation function

$$\varphi_{L,\xi}(A^+(f)A(g)) = \xi \left\langle g, \frac{L}{1-\xi L} f \right\rangle \tag{5}$$

Here $\xi > 0$ is a small positive number and L is a bounded positive operator in $\mathcal{H}_1$ commuting with the one-particle free evolution $S_t = e^{itH_1}$ [the multiplication operator by a function $L(k)$]. In the case $L = \mathrm{e}^{-\beta H_1}$, so that $L(k) = \mathrm{e}^{-\beta\omega(k)}$, the state $\varphi_{L,\xi}$ is just the Gibbs state with the two point correlation function

$$\varphi_{L,\xi}(a^+(k)a(k')) = n(k)\delta(k-k')$$

where $n(k)$ is the density of particles of the gas with momentum k:

$$n(k) = \frac{\xi e^{-\beta k^2/2m}}{1-\xi e^{-\beta k^2/2m}}$$

Notice that in the limit $\xi \to 0$ the density goes to zero. Therefore the limit $\xi \to 0$ is equivalent to the limit $n(k) \to 0$.

The dynamics of the total system is determined by the evolution operator which in interaction representation has the form $U(t) := \mathrm{e}^{\mathrm{i}tH_{\mathrm{free}}}\mathrm{e}^{-\mathrm{i}tH_{\mathrm{tot}}}$. The evolution operator satisfies the differential equation

$$\frac{dU(t)}{dt} = -\mathrm{i}H_{\mathrm{int}}(t)U(t),$$

where $H_{\mathrm{int}}(t) = \mathrm{e}^{\mathrm{i}tH_{\mathrm{free}}}H_{\mathrm{int}}\mathrm{e}^{-\mathrm{i}tH_{\mathrm{free}}}$ is the free evolution of the interaction Hamiltonian. The iterated series for the evolution operator is

$$U(t) = 1 + \sum_{n=1}^{\infty}(-\mathrm{i})^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n H_{\mathrm{int}}(t_1)\ldots H_{\mathrm{int}}(t_n) \tag{6}$$

With the notation $D(t) := \mathrm{e}^{\mathrm{i}tH_{\mathrm{S}}}D\mathrm{e}^{-\mathrm{i}tH_{\mathrm{S}}}$ the evolved interaction becomes

$$H_{\mathrm{int}}(t) := D(t)\otimes A^+(S_tg_0)A(S_tg_1) + D^+(t)\otimes A^+(S_tg_1)A(S_tg_0).$$

Using the spectral decomposition (4) and introducing the set of all Bohr frequencies B, that is the spectrum of the free test particle's Liouvillean $i[H_{\mathrm{S}},\cdot]$, one can write the free evolution of D as

$$D(t) = \sum_{\omega\in B} D_\omega e^{-it\omega}; \qquad D_\omega = \sum_{k,m:\varepsilon_m-\varepsilon_k=\omega} P_kDP_m$$

The reduced dynamics of any test particle's observable X in the low density limit is defined as the limit

$$T_t(X) := \lim_{\xi\to 0}\varphi_{L,\xi}(U^+(t/\xi)(X\otimes 1)U(t/\xi))$$

where $\varphi_{L,\xi}(\cdot)$ denotes partial expectation. The reduced density matrix $\rho(t)$ is defined through the duality $\mathrm{Tr}\,(\rho(0)T_t(X)) = \mathrm{Tr}\,(\rho(t)X)$. As it was mentioned in the Introduction, in the white noise approach the generator of the limiting semigroup can be easily derived from the quantum white noise equation.

3 The White Noise Approach

In Refs. [2, 3, 9] the dynamics of the total system is constructed in the Fock-antiFock representation for the CCR algebra, which is unitary equivalent to the GNS representation. It is defined as follows.

Denote by $\mathcal{H}_1^\iota$ the conjugate of $\mathcal{H}_1$, i.e. $\mathcal{H}_1^\iota$ is identified to $\mathcal{H}_1$ as a set and the identity operator $\iota: \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is antilinear: $\forall f \in \mathcal{H}_1$, $c \in \mathbb{C}$

$$\iota(cf) = c^* \iota(f), \qquad \langle \iota(f), \iota(g) \rangle_\iota = \langle g, f \rangle$$

Then, $\mathcal{H}_1^\iota$ is a Hilbert space and, if the vectors of $\mathcal{H}_1$ are thought as ket-vectors $|\xi\rangle$, then the vectors of $\mathcal{H}_1^\iota$ can be thought as bra-vectors $\langle\xi|$. The corresponding Fock space $\Gamma(\mathcal{H}_1^\iota)$ is called the anti-Fock space.

In this section we assume that for any $t \in \mathbb{R}$: $\langle g_0, S_t g_1 \rangle = 0$. In this case it was shown in Ref. [2] that the dynamics of the total system is given by the family of unitary operators $U_t^{(\xi)}$ in $\mathcal{H}_S \otimes \Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_1^\iota)$ which satisfy the Schrödinger equation:

$$\partial_t U_t^{(\xi)} = -iH_\xi(t) U_t^{(\xi)}, \qquad U_0^{(\xi)} = 1.$$

Here the part of the modified Hamiltonian, which gives a nontrivial contribution in the low density limit, has the form

$$\begin{aligned} H_\xi(t) \; &= \; \sum_{\omega \in B} D_\omega e^{-it\omega} \otimes \Big\{ A^+(S_t g_0) A(S_t g_1) \otimes 1 \\ &\quad + \sqrt{\xi} \big[A(S_t g_1) \otimes A(S_t e^{-\beta H_1/2} g_0) + A^+(S_t g_0) \otimes A^+(S_t e^{-\beta H_1/2} g_1) \big] \Big\} + h.c. \end{aligned}$$

After the time rescaling $t \to t/\xi$ the evolution operator satisfies the equation

$$\frac{dU_{t/\xi}^{(\xi)}}{dt} = -i \sum_{\omega \in B} \Big\{ D_\omega \otimes \big[N_{0,1,\xi}(\omega, t) + B_{1,0,\xi}(-\omega, t) + B_{0,1,\xi}^+(\omega, t) \big] + h.c. \Big\} U_{t/\xi}^{(\xi)}$$

where we introduce for each $n, m = 0, 1$ and $\omega \in B$ the rescaled fields:

$$N_{n,m,\xi}(\omega, t) \; := \; \frac{1}{\xi} e^{-it\omega/\xi} A^+(S_{t/\xi} g_n) A(S_{t/\xi} g_m) \otimes 1 \tag{7}$$

$$B_{n,m,\xi}(\omega, t) \; := \; \frac{1}{\sqrt{\xi}} e^{it\omega/\xi} A(S_{t/\xi} g_n) \otimes A(S_{t/\xi} e^{-\beta H_1/2} g_m) \tag{8}$$

and $B_{n,m,\xi}^+(\omega, t)$ is the adjoint of $B_{n,m,\xi}(\omega, t)$.

3.1 Master Field

It is convenient to use the energy representation for the investigation of the limit as $\xi \to 0$ of the rescaled fields (7), (8). It is defined in terms of the projections

$$P_E := \delta(H_1 - E)$$

which satisfy the properties

$$P_E P_{E'} = \delta(E - E')P_E, \qquad P_E^* = P_E, \qquad S_t = \int dE P_E e^{itE}$$

Define the energy representation for the fields (7), (8) as

$$N_{n,m,\xi}(E_1, E_2, \omega, t) \;:=\; \frac{e^{it(E_1 - E_2 - \omega)/\xi}}{\xi} A^+(P_{E_1} g_n) A(P_{E_2} g_m) \otimes 1 \tag{9}$$

$$B_{n,m,\xi}(E_1, E_2, \omega, t) \;:=\; \frac{e^{it(E_2 - E_1 + \omega)/\xi}}{\sqrt{\xi}} A(P_{E_1} g_n) \otimes A(P_{E_2} e^{-\beta H_1/2} g_m) \tag{10}$$

and let $B^+_{n,m,\xi}(E_1, E_2, \omega, t)$ be the adjoint of $B_{n,m,\xi}(E_1, E_2, \omega, t)$. The operators (7),(8) can be expressed in terms of (9),(10) as

$$N_{n,m,\xi}(\omega, t) = \int dE_1 dE_2 N_{n,m,\xi}(E_1, E_2, \omega, t)$$

$$B_{n,m,\xi}(\omega, t) = \int dE_1 dE_2 B_{n,m,\xi}(E_1, E_2, \omega, t)$$

Let Ω be the set of all linear combinations of the Bohr frequencies with integer coefficients: $\Omega = \{\omega \,|\, \omega = \sum_k n_k \omega_k \text{ with } n_k \in \mathbb{Z},\, \omega_k \in \mathrm{B}\}$. Extend the definition of the fields (9),(10) to arbitrary $\omega \in \Omega$. The limit as $\xi \to 0$ of the fields (9),(10) was found in Ref. [9] and is given by the following theorem.

Theorem 1 *The limits of the rescaled fields*

$$X_{n,m}(E_1, E_2, \omega, t) := \lim_{\xi \to 0} X_{n,m,\xi}(E_1, E_2, \omega, t), \qquad X = B, B^+, N$$

exist in the sense of convergence of correlators and satisfy the commutation relations

$$[B_{n,m}(E_1, E_2, \omega, t), B^+_{n',m'}(E_3, E_4, \omega', t')] = 2\pi \delta_{\omega,\omega'} \delta_{n,n'} \delta_{m,m'} \delta(t' - t)$$
$$\times \delta(E_1 - E_3)\delta(E_2 - E_4)\delta(E_1 - E_2 - \omega)\langle g_n, P_{E_1} g_n\rangle \langle g_m, P_{E_2} e^{-\beta H_1} g_m\rangle \tag{11}$$

$$[B_{n,m}(E_1, E_2, \omega, t), N_{n',m'}(E_3, E_4, \omega', t')] = 2\pi \delta_{n,n'} \delta(t' - t)$$
$$\times \delta(E_1 - E_3)\delta(E_1 - E_2 - \omega)\langle g_n, P_{E_1} g_n\rangle B_{m',m}(E_4, E_2, \omega - \omega', t) \tag{12}$$

$$[N_{n,m}(E_1, E_2, \omega, t), N_{n',m'}(E_3, E_4, \omega', t')] = 2\pi \delta(t' - t)$$
$$\times \{\delta_{m,n'} \delta(E_2 - E_3)\delta(E_3 - E_1 + \omega)\langle g_m, P_{E_2} g_m\rangle N_{n,m'}(E_1, E_4, \omega + \omega', t)$$
$$-\delta_{n,m'} \delta(E_1 - E_4)\delta(E_3 - E_1 - \omega')\langle g_n, P_{E_1} g_n\rangle N_{n',m}(E_3, E_2, \omega + \omega', t)\} \tag{13}$$

The causal commutation relations of the master field are obtained replacing in (11)–(13) the factor $\delta(t' - t)$ by $\delta_+(t' - t)$, where the causal δ-function $\delta_+(t' - t)$ is defined in Ref. [1], section (8.4); $2\pi\delta(E_1 - E_2 - \omega)$ by $(i(E_1 - E_2 - \omega - i0))^{-1}$ and $2\pi\delta(E_3 - E_1 \pm \omega)$ by $(i(E_3 - E_1 \pm \omega - i0))^{-1}$.

3.2 Fock Representation of the Master Field

The next step is to realize the algebra of the master field by operators acting in a Hilbert space. We realize it in the Fock space, which is constructed as follows.

Let K be a vector space of finite rank operators acting on the one-particle Hilbert space $\mathcal{H}_1$ with the property that for any $\omega \in \Omega$; $X, Y \in K$

$$\begin{aligned} \langle X, Y\rangle_\omega : &= \int dt \mathrm{Tr}\left(e^{-\beta H_1} X^* S_t Y S_t^*\right) e^{-i\omega t} \equiv \int dt \mathrm{Tr}\left(e^{-\beta H_1} X^* Y_t\right) e^{-i\omega t} \\ &\equiv 2\pi \int dE \mathrm{Tr}\left(e^{-\beta H_1} X^* P_E Y P_{E-\omega}\right) < \infty \end{aligned}$$

where $Y_t = S_t Y S_t^*$ the free evolution of Y. It was shown in Ref. [3] that the space K is non empty and $\langle \cdot, \cdot \rangle_\omega$ defines a prescalar product on K. Let K_ω be the Hilbert space with inner product $\langle \cdot, \cdot \rangle_\omega$ obtained as completion of the quotient of K by the zero $\langle \cdot, \cdot \rangle_\omega$-norm elements. Denote $\mathcal{K} := \bigoplus_{\omega \in \Omega} K_\omega$.

Consider the Fock space

$$\Gamma(L^2(\mathbb{R}_+) \otimes \mathcal{K}) \equiv \Gamma(\bigotimes_{\omega \in \Omega} L^2(\mathbb{R}_+, K_\omega)) \equiv \bigotimes_{\omega \in \Omega} \Gamma(L^2(\mathbb{R}_+, K_\omega)) \tag{14}$$

where the last infinite tensor product is referred to the vacuum vectors. Let $b^+_{t,\omega}(X), b_{t,\omega}(X)$ be the white noise creation and annihilation operators in this Fock space. These operators satisfy the commutation relations

$$[b_{t,\omega}(X), b^+_{t',\omega'}(Y)] = \delta(t'-t)\delta_{\omega,\omega'}\langle X, Y\rangle_\omega.$$

Each white noise operator $b_{t,\omega}(\cdot)$ acts as usual annihilation operator in $\Gamma(L^2(\mathbb{R}_+, K_\omega))$ and as identity operator in other subspaces.

The representation of the algebra (11)–(13) can be constructed in the Fock space (14) by the identification

$$B_{n,m}(E_1, E_2, \omega, t) = b_{t,\omega}(|P_{E_1} g_n\rangle\langle P_{E_2} g_m|)$$

The number operator is defined as

$$\begin{gathered} N_{n,m}(E_1, E_2, \omega, t) = \\ \sum_{\varepsilon=0,1} \sum_{\omega_1 \in \Omega} \mu_\varepsilon(E_1 - \omega_1) b^+_{t,\omega_1}(|g_n\rangle\langle P_{E_1-\omega_1} g_\varepsilon|) b_{t,\omega_1-\omega}(|P_{E_2} g_m\rangle\langle P_{E_1-\omega_1} g_\varepsilon|) \end{gathered}$$

with $\mu_\varepsilon(E) := \langle g_\varepsilon, P_E e^{-\beta H_1} g_\varepsilon\rangle^{-1}$. One easily checks that these operators satisfy the commutation relations (11)–(13). Denote

$$\begin{aligned} N_{n,m}(\omega, t) &:= \int dE_1 dE_2 N_{n,m}(E_1, E_2, \omega, t) \\ B_{n,m}(E, \omega, t) &:= \int dE' B_{n,m}(E', E, \omega, t) \end{aligned}$$

The limiting white noise Hamiltonian acts in $\mathcal{H}_{\mathrm{S}} \otimes \Gamma(L^2(\mathbb{R}_+) \otimes \mathcal{K})$ as

$$H(t) = \sum_{\omega \in B} D_\omega \otimes \left\{ N_{0,1}(\omega, t) + \int dE \Big[B_{1,0}(E, -\omega, t) + B^+_{0,1}(E, \omega, t) \Big] \right\} + h.c.$$

3.3 Quantum White Noise, Langevin and Boltzmann Equations

The white noise Schrödinger equation for the evolution operator in the low density limit is

$$\partial_t U_t = -iH(t)U_t \tag{15}$$

Following the general theory of white noise equations, in order to give a precise meaning to this equation we will put it in the causally normally ordered form, in which all annihilators are put on the right hand side of the evolution operator and all creators are on the left hand side. This procedure gives a normally ordered quantum white noise equation, which is equivalent to a quantum stochastic differential equation.

For any $n,m \in \{0,1\}$, $\omega \in \Omega$, $E \in \mathbb{R}_+$ let $R^{n,m}_{\omega,\omega'}(E)$ be operators in $\mathcal{H}_{\mathrm{S}}$ which are explicitly defined in Sec. 7, Ref. [9]. The following theorem was proved in Ref. [9].

Theorem 2 *The normally ordered form of equation (15) is*

$$\begin{aligned}\partial_t U_t = \sum_{n,m=0,1} \int dE \Big[&\sum_{\omega,\omega'} R^{n,m}_{\omega,\omega'}(E) \sum_{\varepsilon=0,1} \mu_\varepsilon(E) B^+_{n,\varepsilon}(E,\omega,t) U_t B_{m,\varepsilon}(E,\omega',t) \\ &+ \sum_\omega \Big(R^{n,m}_{\omega,0}(E) B^+_{n,m}(E,\omega,t)U_t + R^{m,n}_{0,\omega}(E) U_t B_{n,m}(E,\omega,t)\Big) \\ &+ R^{n,m}_{0,0}(E)\langle g_n, P_E e^{-\beta H_1} g_m\rangle U_t \Big]\end{aligned} \tag{16}$$

The normally ordered equation (16) can be applied to derivation of the quantum Langevin equation for test particle's observables. Let X be any observable of the test particle. The Langevin equation is the equation satisfied by the stochastic flow j_t defined by $j_t(X) \equiv X_t := U^+_t X U_t$.

Theorem 3 *The quantity X_t satisfies the quantum Langevin equation:*

$$\begin{aligned}\dot{X}_t = \sum_{n,m=0,1} \int dE \Big[&\sum_{\omega_1,\omega_2} \sum_\varepsilon \mu_\varepsilon(E) B^+_{n,\varepsilon}(E,\omega_1,t) U^*_t \Theta^{n,m}_{\omega_1,\omega_2}(X) U_t B_{m,\varepsilon}(E,\omega_2,t) \\ &+ \sum_\omega \Big(B^+_{n,m}(E,\omega,t) U^*_t \Theta^{n,m}_{\omega,0}(X) U_t + U^*_t \Theta^{m,n}_{0,\omega}(X) U_t B_{n,m}(E,\omega,t)\Big) \\ &+ \langle g_n, P_E e^{-\beta H_1} g_m\rangle U^*_t \Theta^{n,m}_{0,0}(X) U_t \Big]\end{aligned} \tag{17}$$

where the structure maps are $\Theta^{n,m}_{\omega_1,\omega_2}(X) := X R^{n,m}_{\omega_1,\omega_2}(E) + R^{+m,n}_{\omega_2,\omega_1}(E) X + 2\sum_{\varepsilon,\omega} \mathrm{Re}\gamma_\varepsilon(E+\omega) R^{+\varepsilon,n}_{\omega,\omega_1}(E) X R^{\varepsilon,m}_{\omega,\omega_2}(E)$.

Equation (17) can be written in terms of the stochastic differentials:

$$\begin{aligned}dj_t(X) = j_t \circ \sum_{n,m} \int dE \Big[&\sum_{\omega_1,\omega_2} \Theta^{n,m}_{\omega_1,\omega_2}(X) dN_t(Z^{n,m}_{\omega_1,\omega_2}(E)) \\ + \sum_\omega \Big(&\Theta^{n,m}_{\omega,0}(X) dB^+_t((|g_n\rangle\langle P_E g_m|)_\omega) + \Theta^{m,n}_{0,\omega}(X) dB_t((|g_n\rangle\langle P_E g_m|)_\omega)\Big)\Big] \\ &+ j_t \circ \mathcal{L}(X) dt\end{aligned}$$

where $Z^{n,m}_{\omega_1,\omega_2}(E)$ is a certain operator in $\mathcal{K}$, which is explicitly defined in Ref. [9]; $\mathcal{L}$ is a quantum Markovian generator, which has the form of a generator of a quantum dynamical semigroup[17]:

$$\mathcal{L}(X) = \Psi(X) - \frac{1}{2}\{\Psi(1), X\} + i[H_{\text{eff}}, X]$$

Here $\Psi(X) = 2\pi \sum\limits_{\varepsilon,\varepsilon',\omega} \int dE \langle g_\varepsilon, P_E e^{-\beta H_1} g_\varepsilon\rangle \langle g_{\varepsilon'} P_{E+\omega} g_{\varepsilon'}\rangle R^{+\varepsilon',\varepsilon}_{\omega,0}(E) X R^{\varepsilon',\varepsilon}_{\omega,0}(E)$ is a completely positive map and the effective Hamiltonian $H_{\text{eff}} := \sum_\varepsilon \int dE \langle g_\varepsilon, P_E e^{-\beta H_1} g_\varepsilon\rangle \times (R^{+\varepsilon,\varepsilon}_{0,0}(E) - R^{\varepsilon,\varepsilon}_{0,0}(E))/2i$ is selfadjoint.

The following theorem is a direct consequence of the quantum Langevin equation (17) of Ref. [9].

Theorem 4 *The reduced density matrix satisfies the quantum linear Boltzmann equation*

$$\frac{d\rho(t)}{dt} = \mathcal{L}_*(\rho(t)) \tag{18}$$

where the generator $\mathcal{L}_*$ is the dual to the generator $\mathcal{L}$.

The generator $\mathcal{L}_*$ is the sum of its dissipative and Hamiltonian parts, $\mathcal{L}_*(\rho) = \mathcal{L}_{\text{diss}}(\rho) - i[H_{\text{eff}}, \rho]$. The explicit form can be obtained by direct calculations using the expression above for the generator $\mathcal{L}$ and Eq. (8.6) of Ref [9] for the T operator. Finally it has the following form. Let T be the one-particle T-operator for the scattering of the test particle and one particle of the gas and $T_{n,n'}(k,k') = \langle n,k|T|n',k'\rangle$ be its generic matrix element, where $|n\rangle$ is an eigenvector of H_S with eigenvalue ε_n and k the momentum of one particle of the gas. Denote

$$T_\omega(k,k') := \sum_{m,n:\,\varepsilon_m-\varepsilon_n=\omega} T_{m,n}(k,k')|m\rangle\langle n|$$

Density of particles of the gas is determined by the function $L(k)$ in (5). For the Gibbs state $L(k) = e^{-\beta\omega(k)}$. Other forms of the density correspond to non-equilibrium states of the gas and can be controlled, for example, by filtering. In these notations the dissipative part of the generator is

$$\begin{aligned} \mathcal{L}_{\text{diss}}(\rho) = 2\pi \sum_{\omega\in B} \int dk dk' \delta(\omega(k') - \omega(k) + \omega) L(k) \\ \times \Big[T_\omega(k',k)\rho T^+_\omega(k',k) - \tfrac{1}{2}\Big(T^+_\omega(k',k)T_\omega(k',k)\rho + \rho T^+_\omega(k',k)T_\omega(k',k)\Big)\Big] \end{aligned} \tag{19}$$

In the case the gas is in equilibrium this generator coincides with the generator of the quantum linear Boltzmann equation obtained in Ref. [12].

4 White Noise Approach without Fock-antiFock Representation

The approach to derivation of the quantum white noise equations directly in terms of the correlation functions, without use of the Fock-antiFock representation, was developed in Ref. [10]. In this approach one introduces the notion of causal state and causal time-energy quantum white noise and proves the convergence of chronological correlation functions of operators

$$N_{f,g,\xi}(t) = \frac{1}{\xi} A^+(S_{t/\xi} f) A(S_{t/\xi} g) \tag{20}$$

acting in $\Gamma(\mathcal{H}_1)$ to correlation functions of the time-energy quantum white noise (Theorem 5). This time-energy quantum white noise is a family of creation and annihilation operators, with commutator proportional to δ-function of time and energy [see (22)]. These operators act in a Fock space which, in difference with (14), does not depend on the initial state of the gas $\varphi_{L,\xi}$.

Suppose for simplicity that $D(t) = D$. Then the evolution operator $U(t/\xi)$ after the time rescaling $t \to t/\xi$ satisfies the equation

$$\frac{dU(t/\xi)}{dt} = -i(D \otimes N_{g_0,g_1,\xi}(t) + D^+ \otimes N_{g_1,g_0,\xi}(t))U(t/\xi) \tag{21}$$

Theorem 5 and the causal commutation relations (23) are used to show that the limit as $\xi \to 0$ of the rescaled evolution operator satisfies the causally normally ordered equation (30). That equation is equivalent to the quantum stochastic equation (31) which can be written in Hilbert module notations as (34) and then in terms of the one-particle S-matrix as (36).

As it was stated in the Introduction, in this approach the algebra of the time-energy quantum white noise (22), the quantum Ito table (32) and the quantum stochastic equation for the limiting evolution (36) do not depend on the initial state of the gas. This is different from the approach of Sect. 3, where the commutation relations (11) for the master field, the Hilbert space representation (subsection 3.2) and hence the limiting equation (16) depend on the initial state (through the factor $e^{-\beta H_1}$). Instead, the dependence on the initial state of the gas now is contained in the limiting state φ_L (in the equilibrium $L = e^{-\beta H_1}$) (in the approach of Sect. 3 the state of the master field is the vacuum state). Considering the limiting state φ_L as the conditional expectation [with the property (33)], one can derive the quantum master equation for the reduced dynamics of the test particle which coincides, when restricted to the same model, with the analogous equation following from (17).

4.1 Causal Time-Energy Quantum White Noise

Define the Hilbert space $\mathcal{X}_{\mathcal{H}_1,H_1}$ as the completion of the quotient of the set

$$\left\{ F : \mathbb{R}_+ \to \mathcal{H}_1 \text{ s.t. } ||F||^2 := 2\pi \int \mathrm{d}E \langle F(E), P_E F(E)\rangle < \infty \right\}$$

with respect to the zero-norm elements. The inner product in $\mathcal{X}_{\mathcal{H}_1,H_1}$ is $\langle F, G\rangle = 2\pi \int \mathrm{d}E \langle F(E), P_E G(E)\rangle$. Let $B^+_f(E,t)$, $B_g(E',t')$ be the creation and annihilation operators acting in the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+, \mathcal{X}_{\mathcal{H}_1,H_1}))$ over the Hilbert space $L^2(\mathbb{R}_+, \mathcal{X}_{\mathcal{H}_1,H_1})$ of square integrable functions $f : \mathbb{R}_+ \to \mathcal{X}_{\mathcal{H}_1,H_1}$. These operators (operator-valued distributions) satisfy the canonical commutation relations

$$[B_g(E,t),\, B^+_f(E',t')] = 2\pi\delta(t'-t)\delta(E'-E)\langle g, P_E f\rangle \tag{22}$$

and causal commutation relations

$$[B_g(E,t),\ B_f^+(E',t')] = \delta_+(t'-t)\delta(E'-E)\gamma_{g,f}(E) \tag{23}$$

where $\delta_+(t'-t)$ is the causal δ-function and $\gamma_{g,f}(E) = \int_{-\infty}^{0} dt \langle g, S_t f\rangle e^{-itE}$. The meaning of two different commutators (22) and (23) for the same operators is explained in Ref. [1], Sect. 7. These operators are called *time-energy quantum white noise* due to the presence of $\delta(t'-t)\delta(E'-E)$ in (22).

Define the *white noise number operators* as

$$N_{f,g}(t) = \int dE B_f^+(E,t) B_g(E,t) \tag{24}$$

For any positive bounded operator L in $\mathcal{H}_1$ define the *causal gauge-invariant mean-zero Gaussian state* φ_L by the properties (25)–(28):

$$\text{for n} = 2\text{k} \qquad \varphi_L(B_1^{\epsilon_1}\dots B_n^{\epsilon_n}) = \sum \varphi_L(B_{i_1}^{\epsilon_{i_1}} B_{j_1}^{\epsilon_{j_1}})\dots\varphi_L(B_{i_k}^{\epsilon_{i_k}} B_{j_k}^{\epsilon_{j_k}}) \tag{25}$$

where the sum is taken over all permutations of the set $(1,\dots,2k)$ such that $i_\alpha < j_\alpha$, $\alpha = 1,\dots,k$ and $i_1 < i_2 < \dots < i_k$; $B_m^{\epsilon_m} := B_{f_m}^{\epsilon_m}(E_m, t_m)$ for $m = 1,\dots,n$ are time-energy quantum white noise operators with causal commutation relations (23), and ϵ_m means either creation or annihilation operator;

$$\text{for n} = 2\text{k}+1 \qquad \varphi_L(B_1^{\epsilon_1}\dots B_n^{\epsilon_n}) = 0 \tag{26}$$

$$\varphi_L(B_f(E,t)B_g(E',t')) = \varphi_L(B_f^+(E,t)B_g^+(E',t')) = 0 \tag{27}$$

$$\varphi_L(B_f^+(E,t)B_g(E,t')) = \chi_{[0,t]}(t')\langle g, P_E L f\rangle \tag{28}$$

The "state" φ_L does not satisfy the positivity condition. This is a well-known situation for the weak coupling limit (see Ref. [1]) and is due to the fact that we work with time-ordered, or causal correlators. Therefore it is natural to call such "states" causal states.

Def. 1 *Causal time-energy quantum white noise is a pair* $(B_f^\pm(E,t), \varphi_L)$, *where* $B_f^\pm(E,t)$ *satisfy the causal commutation relations (23) and* φ_L *is a causal gauge-invariant mean-zero Gaussian state.*

Theorem 5 *For any* $n \in \mathbb{N}$ *in the sense of distributions over simplex* $t_1 \geq t_2 \geq \dots \geq t_n \geq 0$ *one has the limit*

$$\lim_{\xi\to 0} \varphi_{L,\xi}(N_{f_1,g_1,\xi}(t_1)\dots N_{f_n,g_n,\xi}(t_n)) = \varphi_L(N_{f_1,g_1}(t_1)\dots N_{f_n,g_n}(t_n))$$

This theorem was proved in Ref. [10].

Remark 1 *This convergence is called convergence in the sense of time-ordered correlators. The fact that we use the distributions over simplex is motivated by iterated series (6) for the evolution operator.*

Remark 2 *The proof is based on the fact that for any $n \in \mathbb{N}$ only one connected diagram survives in the limit. This can be interpreted as emergence of a new statistics (different from Bose) in the low density limit*[1]*. For a discussion of new statistic arising in the weak coupling limit see Ref. [1].*

The following theorem is important for investigation of the limiting white noise equation for the evolution operator.

Theorem 6 *The limit state φ_L has the following factorization property:* $\forall n \in \mathbb{N}$

$$\begin{aligned} \varphi_L \quad & (B^+_f(E,t)N_{f_1,g_1}(t_1)\dots N_{f_n,g_n}(t_n)B_g(E,t)) \\ = \quad & \varphi_L(B^+_f(E,t)B_g(E,t))\varphi_L(N_{f_1,g_1}(t_1)\dots N_{f_n,g_n}(t_n)) \end{aligned}$$

where the equality is understood in the sense of distributions over simplex $t \geq t_1 \geq t_2 \geq \dots \geq t_n \geq 0$.

Theorem 5 allows us to calculate the partial expectation of the evolution operator and Heisenberg evolution of any system observable in the low density limit. In fact, partial expectation of the n-th term of the iterated series (6) (or equivalent series for Heisenberg evolution of a system observable) after time rescaling $t \to t/\xi$ includes the quantity

$$\int\limits_0^t \mathrm{d}t_1 \dots \int\limits_0^{t_{n-1}} \mathrm{d}t_n \varphi_{L,\xi}(N_{f_1,g_1,\xi}(t_1)\dots N_{f_n,g_n,\xi}(t_n))$$

(where f_α, g_α are equal to g_0 or g_1). The limit as $\xi \to 0$ of this quantity can be calculated using Theorem 5. For example, the contribution of the connected diagram is equal to

$$\begin{aligned} \int\limits_0^t \mathrm{d}t_1 \int\limits_0^{t_1} \mathrm{d}t_2 \delta_+(t_2 - t_1) \int\limits_0^{t_2} \mathrm{d}t_3 \delta_+(t_3 - t_2) \dots \int\limits_0^{t_{n-1}} \mathrm{d}t_n \delta_+(t_n - t_{n-1}) & \\ \times \int \mathrm{d}E \langle g_n, P_E L f_1 \rangle \gamma_{g_1,f_2}(E) \dots \gamma_{g_{n-1},f_n}(E) & \\ = t \int \mathrm{d}E \langle g_n, P_E L f_1 \rangle \gamma_{g_1,f_2}(E) \dots \gamma_{g_{n-1},f_n}(E) & \end{aligned}$$

Similarly one can calculate the contribution of nonconnected diagrams (they give terms proportional to higher powers of t). Summation over all orders of the iterated series gives the reduced dynamics of the test particle. An advantage of the white noise approach is that it allows to get the limiting dynamics in a nonperturbative way, without direct summation of the iterated series. This procedure includes derivation of the causally normally ordered white noise equation for the limiting evolution operator. After that the reduced dynamics of the test particle can be easily found.

[1] In Ref. [18] the statistics which appears in the low density limit was found and a connection with Voiculescu free independence theory was established.

4.2 The White Noise and Quantum Stochastic Equations

The limiting evolution operator satisfies the white noise Schrödinger equation

$$\frac{\mathrm{d}U_t}{\mathrm{d}t} = -\mathrm{i}(D \otimes N_{g_0,g_1}(t) + D^+ \otimes N_{g_1,g_0}(t))U_t, \qquad U_0 = 1 \tag{29}$$

The next step is to put this equation to the causally normally ordered form, i.e., to put all annihilation operators, appearing in $N_{f,g}(t)$, on the right side of the evolution operator and all creation operators on the left side.

Assume that for each $E \in \mathbb{R}$ the following inverse operators exist

$$T_0(E) := \Big(1 + \gamma_{g_0,g_1}(E)D^+ - \gamma_{g_1,g_0}(E)D + (\gamma_{g_0,g_0}\gamma_{g_1,g_1} - \gamma_{g_1,g_0}\gamma_{g_0,g_1})(E)DD^+\Big)^{-1}$$

$$T_1(E) := \Big(1 + \gamma_{g_0,g_1}(E)D^+ - \gamma_{g_1,g_0}(E)D + (\gamma_{g_0,g_0}\gamma_{g_1,g_1} - \gamma_{g_1,g_0}\gamma_{g_0,g_1})(E)D^+D\Big)^{-1}$$

Denote

$$\begin{aligned} R_{0,0}(E) &:= \gamma_{g_1,g_1}(E)DT_1(E)D^+, \quad R_{0,1}(E) := -DT_1(E)(1 + \gamma_{g_0,g_1}(E)D^+) \\ R_{1,1}(E) &:= \gamma_{g_0,g_0}(E)D^+T_0(E)D, \quad R_{1,0}(E) := D^+T_0(E)(1 - \gamma_{g_1,g_0}(E)D) \end{aligned}$$

Theorem 7 *The causally normally ordered form of equation (29) is*

$$\frac{\mathrm{d}U_t}{\mathrm{d}t} = -\sum_{n,m=0,1} \int \mathrm{d}E R_{m,n}(E) B^+_{g_m}(E,t) U_t B_{g_n}(E,t) \tag{30}$$

This theorem was proved in Ref. [10].

Remark 3 *An immediate consequence of Theorem 6 is the following factorization property of the limiting state φ_L:*

$$\varphi_L(B^+_f(E,t)U_tB_g(E,t)) = \varphi_L(B^+_f(E,t)B_g(E,t))\varphi_L(U_t)$$

This property of the state φ_L similar to the factorization property of the state determined by a coherent vector $\Psi, \|\Psi\| = 1$:

$$(\Psi, B^+_f(E,t)U_tB_g(E,t)\Psi) = (\Psi, B^+_f(E,t)B_g(E,t)\Psi)(\Psi, U_t\Psi)$$

which is usually used to define quantum stochastic differential equations.

Normally ordered white noise equation (30) equivalent through identification

$$B^+_m(E,t)U_tB_n(E,t)\mathrm{d}t = 2\pi \mathrm{d}N_t(|P_Eg_m\rangle\langle P_Eg_n|)U_t$$

to the quantum stochastic differential equation

$$\mathrm{d}U_t = -2\pi \sum_{n,m=0,1} \int \mathrm{d}E R_{m,n}(E)\mathrm{d}N_t(|P_Eg_m\rangle\langle P_Eg_n|)U_t \tag{31}$$

where N_t is the quantum number process in $\Gamma(L^2(\mathbb{R}_+)\otimes\mathcal{H}_1)$. The stochastic differential $\mathrm{d}N_t$ satisfies the quantum Ito table

$$\mathrm{d}N_t(X)\mathrm{d}N_t(Y) = \mathrm{d}N_t(XY) \tag{32}$$

where X, Y are operators in $\mathcal{H}_1$. The limiting state φ_L has the property

$$\varphi_L(2\pi \mathrm{dN}_t(|P_E f\rangle\langle P_E g|)) = \langle g, P_E L f\rangle \mathrm{d}t \tag{33}$$

Equation (31) can be written in Hilbert module notations as

$$\mathrm{d}U_t = \mathrm{d}N_t\Big(-2\pi \sum_{n,m=0,1} \int \mathrm{d}E R_{m,n}(E) \otimes |P_E g_m\rangle\langle P_E g_n|\Big) U_t \tag{34}$$

The one-particle S-matrix for scattering of the test particle on one particle of the gas has the form

$$S = 1 - 2\pi \sum_{n,m=0,1} \int \mathrm{d}E R_{m,n}(E) \otimes |P_E g_m\rangle\langle P_E g_n| \tag{35}$$

This is a unitary operator: $S^+S = SS^+ = 1$. An immediate conclusion from (34) and (35) is the following theorem which was proved in Ref. [10] and is one of the main results of the paper.

Theorem 8 *The evolution operator in the low density limit satisfies the quantum stochastic equation driven by the quantum number process with strength $S-1$:*

$$\mathrm{d}U_t = \mathrm{d}N_t(S-1)U_t \tag{36}$$

This equation can be applied to derivation of the quantum master equation for the reduced dynamics. Let $X \in B(\mathcal{H}_{\mathrm{S}})$ be an observable of the test particle. Its time evolution $X_t = U_t^+ X U_t$ satisfies the equation

$$dX_t = dU_t^+ X U_t + U_t^+ X dU_t + dU_t^+ X dU_t$$

(we identify X and $X \otimes 1$). Now, since

$$dU_t^+ = U_t^+ dN_t(S^+ - 1)$$

(we use the Hilbert module notations, hence $dN_t(S^+ - 1)$ does not commute with U_t^+) and using the quantum Ito table (32) one gets

$$\begin{aligned} dX_t &= U_t^+ dN_t(S^+ - 1) X U_t + U_t^+ X dN_t(S-1) U_t \\ &\quad + U_t^+ dN_t(S^+ - 1) X dN_t(S-1) U_t = U_t^+ dN_t(\Theta(X)) U_t \end{aligned}$$

where the map $\Theta : B(\mathcal{H}_{\mathrm{S}}) \to B(\mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_1)$ has the form $\Theta(X) = (S^+ - 1)X(S-1) + (S^+ - 1)X + X(S-1) \equiv S^+XS - X$. Simple computations, together with the explicit form (35) for the S-matrix and the property $P_E P_{E'} = \delta(E-E')P_E$ of the projector, give the expression

$$\Theta(X) = 2\pi \sum_{n,m=0,1} \int dE \Theta_E^{n,m}(X) \otimes |P_E g_m\rangle\langle P_E g_n|$$

where $\Theta_E^{n,m}(X) = 2\pi \sum\limits_{n',m'} R^+_{n',m}(E) X R_{m',n}(E) \langle g_{n'} P_E g_{m'}\rangle - R^+_{n,m}(E) X - X R_{m,n}(E)$. Therefore

$$dX_t = 2\pi \sum_{n,m=0,1} \int dE \mathrm{d}N_t(|P_E g_m\rangle\langle P_E g_n|) U_t^+ \Theta_E^{n,m}(X) U_t$$

The reduced dynamics $\overline{X}_t := \varphi_L(X_t) \in B(\mathcal{H}_{\mathrm{S}})$ is obtained by taking conditional expectation of both sides of this equation in the state φ_L and using (33):

$$\frac{d\bar{X}_t}{dt} = \sum_{n,m=0,1} \int dE \langle g_n, P_E L g_m \rangle \overline{\Theta_E^{n,m}(X)}_t$$

When restricted to the case of orthogonal form-factors g_0, g_1, this master equation coincides with the one of Ref. [9]. The linear Boltzmann equation for the reduced density matrix is the dual to this equation and has the form of Eq. (18) with dissipative generator of the form Eq. (19).

Appendix[2]

This paper presents two versions of the white noise approach to the investigation of the dynamics of a quantum system interacting with a gaseous environment. The approach allows to describe the dynamics of the total system consisting of the particle and the environment in the low density limit. The low density limit corresponds to the kinetic regime when only pair collisions (i.e., collisions of the test particle at one time moment with one particle of the gas) contribute to the dynamics such that the probabilities of multi particle collisions are negligible. Although the main result of the approach is the quantum stochastic differential equation for the total dynamics, quantum master equation for the reduced system dynamics is derived as its simple consequence [Eq. (18) with Lindblad-GKS generator given by Eq. (19)]. The derivation is performed *ab initio* for general, including non-equilibrium, gases.

Section 2 of the present paper considers cases of discrete and continuous spectrum of the test particle. The limiting equations are explicitly derived here for the case of discrete spectrum. This corresponds to confining the test particle in a spatially finite region, which however can be arbitrarily large. The master equation (18) describing the dynamics of a particle interacting with a gas recently was applied to the problem of non-unitary quantum control [19], where it was used to analyze capabilities of controlling quantum systems by optimizing with learning control algorithms the state of the surrounding gas (i.e., by optimizing its distribution function). Master equations for a quantum Brownian particle in a free space (i.e., with translation invariant dynamics) also attract attention of the researchers. In this context I mention recent works by Bassano Vacchini [20], Klaus Hornberger [21], and Stephen Adler [22], where also other relevant references can be found. The two characteristic features of the white noise approach are that it allows to derive solvable equations for the dynamics of the total system (the test particle and the environment) and that these equations are not phenomenological, they are derived from the exact microscopic dynamics.

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[2]This Appendix was added after publishing the paper to briefly outline some new results in the field.

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